LINEAR PROGRAMMING

Linear programming refers to a mathematical programming technique used to balance many variables so as to achieve a pre-determined objective.

For example

Linear programming plays a key important role in ensuring optimal utilization of such limited resources so that the business can achieve its objective. Linear programming can be used to balance production capacity so as to minimize costs and maximize profits.

In order to apply linear programming there must be linear relationship between the factors under condition, where some of these factors are variable and the others are not variable.

Linear programming is developed from a wider mathematical technique called programming.

Programming refers to the mathematical technique used to get the best possible solution to problem involving limited or scarce resources.

Areas where linear programming can be applied:

1. Transportation problem
2. Investment decision
3. Project mix problem
4. Job assignment
5. Production scheduling
6. Purchasing
7. Capacity allocation problem
8. Location problem.etc

Limitations of linear programming

1. Assumes that all variables are linearly related.
2. Assumes that all projects are divisible.
3. Uncertainty is ignored.
4. Assumes that the returns of project are directly proportional to the amount invested in each project.
5. Assumes that linear programming does not work for mutually exclusive events.
6. Linear programming assumes that project and constraints are independent.

Linear programming problems

A linear programming problem is one in which we are to find the maximum or minimum value of a linear expression such as

$ax + by + cz +$ ....(objective function)

Linear constraints

$Ax + By + Cz$ .... $\leq N$ (Maximisation inequality)

$Ax + By + Cz$ .... $\geq N$ (Minimisation inequality)

$x \geq 0, y \geq 0$ .... (Non-negativity)

Features of linear programming problems

1. The problem must be capable of being stated in mathematical terms.
2. All factors involved in the problem must have linear relationship e.g. to double output, we have to double labour output.
3. The problem must permit choices between alternative courses of action.
4. There must be one or more restrictions on the factors involved e.g. restrictions on labour and materials.

Requirements for formulating a linear programming problem

a) There must be an objective function to either the objective must be profit maximization or cost minimization.

b) There must be resources constraints.

c) We should be able to express the objective function and constraints in the mathematical form.

d) We assume existence of a non-negativity condition.

e) All the objective function and the constraints must be a linear equation of degree one.

Terms under linear programming

A variable; this refers to a value that changes as a result of an action taken by the manager.

Decision variable; this represents the amount or number of each product that the decision maker should produce in order to achieve his or her objective.

A solution; this refers to a set of values for the given variable.

A constraint; this refers to a limitation of the required decision variable

A constraint function; the portion of the constraint containing the variable.

Feasible solution; a solution that satisfies all the constraints simultaneously.

Feasible region; this refers to a set of all possible solutions.

Feasible solution point; is any point on the boundary of the feasible region.

Objective function; this is a function that shows the goal to be achieved by the firm that is to say maximum profit/ minimum cost.

Non-negativity constraints; a set of constraints that require variables to be greater or equal to zero never be negative e.g. $x \geq 0$; $y \geq 0$.

Iso-profit line; this refers to the line in the feasible region on which different combinations produce the same profit/ maximum profit.

Iso-cost line; this is a line in the feasible solution. A basic solution that also satisfies all the non-negativity condition and all variables are always greater or equal to zero.

Optimum solution; this is a feasible solution that maximizes/ minimizes the value of the company's objective function.

Multiple optimal solution; this is an objective function attains its optimum value at more than one feasible point.

Infeasibility; this is where there are no solutions to the given problem satisfying all the constraints including the non-negativity conditions.

Unboundness; this is a solution where the value of the objective function can be increased without limit.

Slack; this is the amount of a scarce resource that is unused by the given optimal mix or solution in any linear programming model.

It can range from zero for the case in which all the scarce resources are utilized to the original amount of scarce resource that was available.

The term slack is used for maximization problems and where there less than signs.

Uses of slack

The excess resources can be used for;

1. Hiring out of the machinery to generate more revenue fro the company.
2. Maintaining the machines.
3. Producing another product.
4. Using the time to conduct seminars and other training sessions.
5. Sending the workers on leave during the slack period.

Formulating linear programming models

The following steps are followed.

i) Summarize the given information in form of a table or same times the data/ information can be given when it's already summarized as below.

Products	Activities	Objectives
X		
Y		
Total of the resources available		

ii) From table, write down the objective function as a linear equation and the resource constraints as linear inequality.

iii) Sketch the lines relating to the inequalities above by first replacing the inequality signs with equal signs hence determine the coordinates of each equation and put them.

iv) For each line determine the wanted and the unwanted region.
This is done by picking on either side of the line and substituting it the corresponding inequality.
If the inequality is satisfied then the region from which it was chosen is wanted and the other one is unwanted. We usually shade off the unwanted region.

v) The corner points of the unshaded region (feasible region) become the decision making points. They are then used to find at which one of them is the business meeting it is objective.

Methods of solving linear programming problems

There are two methods of solving linear programming problems and these include:

1. Graphical method
2. Simplex method

Graphical Method

This method involves in solving the problem equations with two unknown, which involves by determining the coordinates and are plotted on the x-axis & y-axis and are being joined them together. Considering either maximum or minimum of inequality while shading the graph.

Question 1

The table below shows the company's two products x and y. The products passes through assembling (hours) and finishing process and the profit should be maximized.

Products	Assembling hours	Finishing hours	Profit per unit
X	4	2	8
Y	2	4	6
Total hours	60	48	

Required: Formulate the;

i) Objective function to maximize profit. (2 marks)

ii) Inequality constraints. (3 marks)

iii) Non-negativity conditions/ constraints. (2 marks)

iv) Using graphical method: indicate the feasible region. (5 marks)

v) Determine the optimum solution point. (5 marks)

vi) Determine the maximum profit. (4 marks)

vii) Determine the Iso - profit line. (2 marks)

viii) Determine whether the slack is existing or not. (2 marks)

Solution

i) Objective function

Profit = $8x + 6y$

ii). Inequality constraints

Assembling (hours): $4x + 2y \leq 60$

Finishing (hours): $2x + 4y \leq 48$

iii) Non-negativity condition/ constraints.

$x \geq 0, y \geq 0$

iv) Using graphical method to indicate the feasible region.

- We first change the inequality into equations.
- We obtain the coordinates
- Then we plot them to determine the feasible region as follows

Inequality	equations
$4x + 2y \leq 60$;	$4x + 2y = 60$
$2x + 4y \leq 48$;	$2x + 4y = 48$

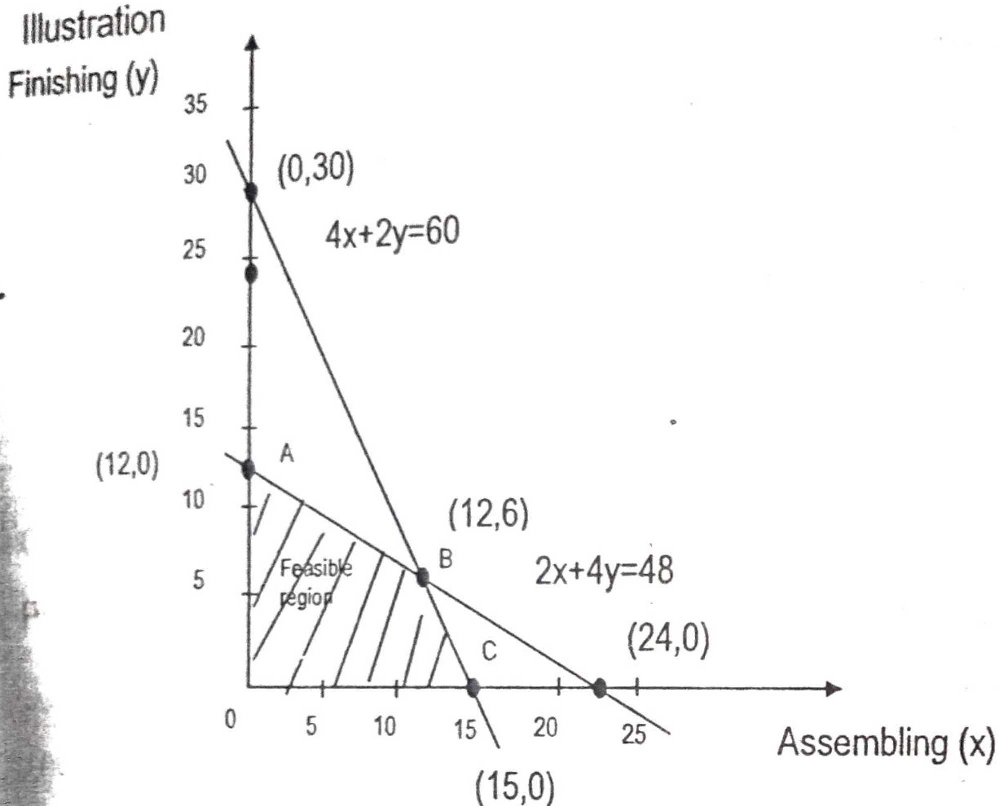
We obtain coordinates from each equation.

$4x + 2y = 60$: $2x + 4y = 48$

X	0	15
Y	30	0

x	0	24
y	12	0

Illustration



v) Optimum solution point

Here we solve the two equations simultaneously or we can obtain it direct from the graph in case you have used graph paper to plot the equations to be accurate.

Therefore:

$4x + 2y = 60$ ....(i)

$2x + 4y = 48$ ....(ii)

Dividing eq(i) by 2 gives

Using subtraction method

$2x + y = 30$

$\underline{-2x + 4y = 48}$

$0 + -3y = -18$

$$\frac{-3y}{-3} = \frac{-18}{-3}$$

$$y = 6$$

Substituting the value of y in any equation gives

2x + 4(6) = 48

2x + 24 = 48

2x = 48 – 24

$$\frac{2x}{2} = \frac{24}{2}$$

x = 12

Comparing the points which bounded off the feasible region.

A(0,12)

B(12,6)

C(15,0)

Therefore the optimum solution point is B(12,6)

vi) Maximum profit

Profit = 8x + 6y

Here we consider the points A(0,12), B(12,6) and C(15,0) the one which will yield the highest value it will be the one to maximize the profit. (Because the project aims at higher value for maximization of the profit).

Consider; Profit = 8x+ 6y

Consider point A(0,12) substituting in the above equation gives

Profit = 8(0) + 6(12)

= 0 + 72 = 72

Consider point B(12,6) substituting in the above equation gives

Profit = 8(12) + 6(6)

= 96 + 3 = 132

Consider point C(15,0) substituting in the above equation gives

Profit = 8(15) + 6(0)

= 120 + 0 = 120

Therefore profit is maximized at B(12, 6) which yield profit = Shs.132

vii) Iso-Profit line

We consider objective function by equating it to maximum profit.

Therefore 8x + 6y = profit

Where profit = Shs.132

Substituting in the above equation gives

8x + 6y = 132

6y = 132 – 8x

$$\frac{6y}{6} = \frac{132}{6} - \frac{8x}{6}$$

$$y = 22 - 1\frac{1}{3}x$$

It is the required Iso-profit line.

viii) Slack

To get the slack, we substitute the optimal solutions in the constraints.

You realize that the one constraint can be exhausted.

Consider the constraints

$4x + 2y \leq 60$ .... (i); optimum point (12,6)

$4(12) + 2(6) \leq 60$

$48 + 12 \leq 60$

60=60, no slack

$2x + 4y \leq 48$; optimum point (12, 6)

$2(12) + 4(6) \leq 48$

$24 + 24 \leq 48$

48 = 48, no slack

Therefore, there was no slack, meaning that all the resources were utilized at the maximum.

Question 2

A firm produces two types of widgets, manual and electric. Each requires in its manufacture the use of 3 machines; A, B and C. the production process use up time as in the following table.

Widgets	A	B	C	Profit/Units
Manual	2	1	1	4
Electric	1	2	1	6
Hours available	180	160	100	

Required: Determine the;

i) Objective function to maximize the profit. (2 marks)

ii) Inequality constraints. (4 marks)

iii) Non-negativity condition/ constraints. (1 mark)

iv) Using the graphical method show the feasible region. (5 marks)

v) Optimum solution point. (5 marks)

vi) Maximum profit. (4 marks)

vii) Iso-profit line. (2 marks)

viii) Slack

Solution

Let manual = x

Electric = y

i) Objective function

Profit = 4x + 6y

ii) Inequality constraints

A: $2x + y \leq 180$

B: $x + 2y \leq 160$

B: $x + y \leq 100$

iii) Non-negativity condition/ constraints

$x \geq 0; y \geq 0$

iv) Illustration

Here we change the constraints into equations and we determine the coordinates.

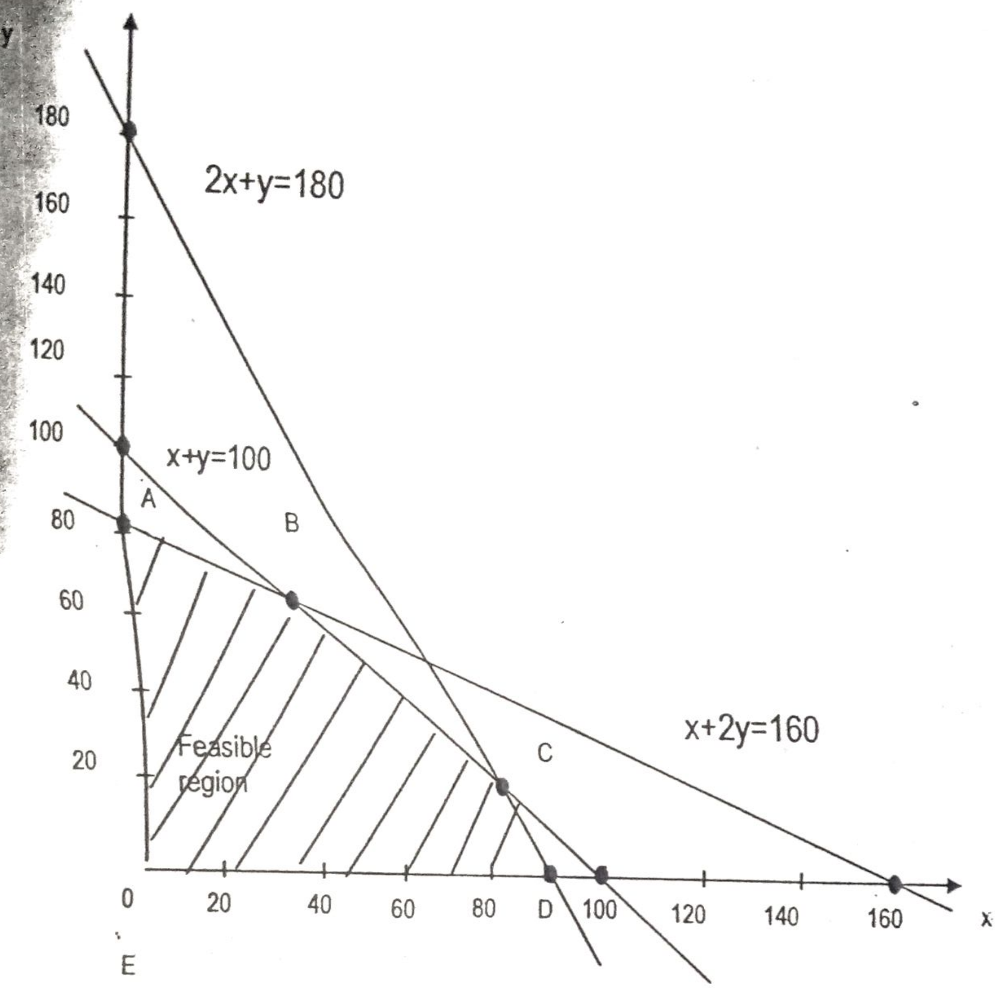
A: $2x + y = 180$; B: $x + 2y = 160$; C: $x + y = 100$

X	0	90
Y	180	0

x	0	160
y	80	0

x	0	100
y	100	0

Graph



v) Optimum solution point

The points bounded the feasible solution gives the following coordinates.

A(0,80)

B: ( )

$x + 2y = 160$

$x + y = 100$

We solve the above equation simultaneously to obtain the coordinates of B.

$$\begin{array}{r} x + 2y = 160 \\ \underline{-x + y = 100} \\ 0 + y = 60 \end{array}$$

y = 60 substituting in any equation gives

x + 60 = 100

x = 100 – 60

x = 40

B(40,60)

C(   )

2x + y = 180

x + y = 100

We solve the above equation simultaneously to obtain the coordinates of C.

2x + y = 180

-x + y = 100

x + 0 = 80

x = 80: substituting in any equation gives

80 + y = 100

y = 100 – 80

y = 20

C(80,20)

D(90,0)

E(0,0)

Optimum solution point is B.

vi) Profit maximization

A, B, C, D, and E

(0,80) (40,60),(80,20), (90,0) (0,0)

Objective function.

P = 4x + 6y

A(0,80)

P = 4(0) + 6(80)

= 0 + 480

P = 480

C(80,20)

P = 4(80) + 6(20)

= 320 + 120

P = 440

B(40,60)

P = 4(40) + 6(60)

= 160 + 360

P = 520

D(90,0)

P = 4(90) + 6(0)

= 360 + 0

P = 360

E(0,0)

P = 4(0) + 6(0)

= 0 + 0

P = 0

Therefore profit is maximized at the point B(40,60) which yield Shs.520.

vii) Iso-profit line

$4x + 6y = \text{profit}$

Where profit = 520

Therefore, $4x + 6y = 520$

$6y = 520 - 4x$

$$\frac{6y}{6} = \frac{520}{6} - \frac{4x}{6}$$

$Y = 86.7 - 0.7x$

It is the required Iso-profit line.

viii) Slack

Considering the constraints

A: $2x + y \leq 180$

B: $x + 2y \leq 160$

C: $x + 2y \leq 100$

Considering optimum solution point at B(40,60)

At A: $2(40) + 60 < 180$

$80 + 60 < 180$

$140 < 180$

Therefore slack is 180 – 140 = 40 Hours remained unutilized, which can be used for other activities.

At B: $40 + 2(60) = 160$

$40 + 120 = 160$

$160 = 160$

No slack, means that all hours were utilized.

At C: $40 + 60 = 100$

$100 = 100$

No slack, means that all hours were utilized.

Question 3

An automobile manufacturing firm produces only trucks and cars for which it uses only three inputs-labour, machine and steel. The firm gets the contractual supplies of inputs and per agreement, it requires to make use of a minimum quantity of inputs of 160 man-hours, 36 machine hours and 48 tones of steel. Input requirements per truck are 40 man hours, 3 machine hours and 8 tones of steel, while input requirements per car are 10 men-hours, 6 machine hours and 4 tones of steel. Given that the production cost per truck has been at $60,000 and the cost per car is $20,000. Formulate an appropriate linear programming model and determine the combination of trucks and cars the firm should manufacture to minimize costs. (15 marks)

Solution

Summary of the above information

	Labour	Machine	Steel	Costs
Trucks	40	3	8	60,000
Cars	10	6	4	20,000
Total hours	160	36	48	

Let Truck = x

Car = y

Objective function

Cost = 60,000x + 20,000y

Constraints

Labor: $40x + 10y \geq 160$

Machine: $3x + 6y \geq 36$

Steel: $8x + 4y \geq 48$

Non-negativity

$x \geq 0, y \geq 0$

Change the inequality constraint into equation to obtain the coordinates.

Labour

$40x + 10y = 160$

x	0	4
y	16	0

Machine

$3x + 6y = 36$

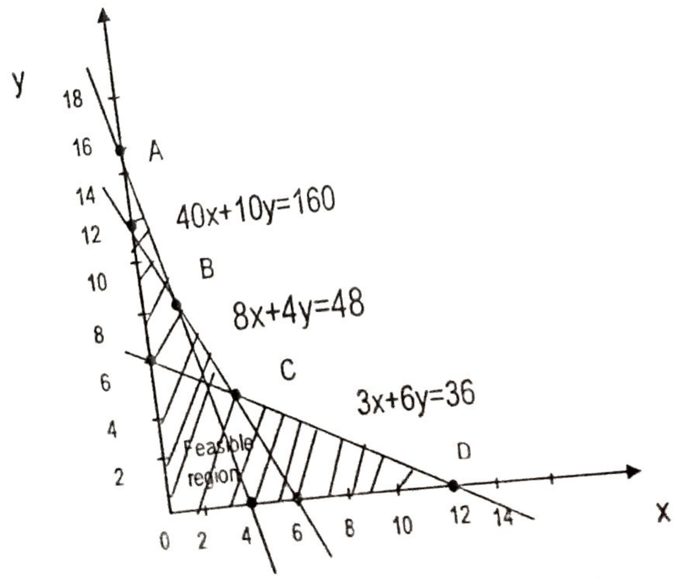
X	0	12
Y	6	0

Steel

$8x + 4y = 48$

x	0	6
y	12	0

Illustration



The combination of truck and cars occurs at:

B: $40x + 10y = 160$

$8x + 4y = 48$

C: $3x + 6y = 36$

$8x + 4y = 48$

Solving the above equation simultaneously to obtain coordinates of point B and C.

Considering point B

Therefore $40x + 10y = 160$ ....(i)

$8x + 4y = 48$ ....(ii)

Divide eq(i) by 10 and eq(ii) by 4 to eliminate y.

Using subtraction method

$4x + y = 16$

$\underline{-2x + y = 12}$

$2x + 0 = 4$

$\frac{2x}{2} = \frac{4}{2}$

$x = 2$

Substituting x in either eq(i) or (ii) to find y gives

$40(2) + 10y = 160$

$80 + 10y = 160$

$10y = 160 - 80$

$\frac{10y}{10} = \frac{80}{10}$

$y = 8$

B(2,8) coordinates

Considering point C

$3x + 6y = 36$ ....(i)

$8x + 4y = 48$ ....(ii)

Divide eq(i) by 3 and eq(ii) by 8 to eliminate x.

Using subtraction method

$x + 2y = 12$

$\underline{-x + 0.5y = 6}$

$0 + 1.5y = 6$

$$\frac{1.5y}{1.5} = \frac{6}{1.5}$$

$y = 4$

Substituting the value of y in any equation to find x gives

$3x + 6(4) = 36$

$3x + 24 = 36$

$3x = 36 - 24$

$$\frac{3x}{3} = \frac{12}{3}$$

$x = 4$

C(4,4) coordinates

The cost at point B is.

Using objective function

Cost = 60,000x + 20,000y

B(2,8) substituting in the cost equation gives.

Cost = 60,000(2) + 20,000(8)

Cost = 120,000 + 160,000

Cost = $280,000

The cost at point C is

Using objective function

Cost = 60,000x + 20,000y

C(4,4) substituting in the cost equation gives.

Cost = 60,000(4) + 20,000(4)

Cost = 240,000 + 80,000

Cost = $320,000

Therefore the combination of trucks and cars the firm should manufacture to minimize costs are 2 trucks and 8 cars at the cost of $280,000

Note: therefore iso- cost line

Here consider objective function equating it to minimum cost

60,000x + 20,000y = cost

60,000x + 20,000y = 280,000

20,000y = 280,000 – 60,000x

$$\frac{20{,}000y}{20{,}000} = \frac{280{,}000}{20{,}000} - \frac{60{,}000x}{20{,}000}$$

y = 14 – 3x

The iso-cost line shows the all combinations that yield a minimum cost of $280,000.

Question 4

A Cereal manufacturer wants to make a new brand of cereal combining two natural gains y_1 and y_2. The new cereal must have a minimum of 128 units of carbohydrates, 168 units of protein and 120 units of fructose. Gain 1 has 24 units of carbohydrates, 14 units of proteins and 8 units of fructose. Grain 2 has 4 units of carbohydrates, 7 units of protein and 32 units of fructose. Grain 1 cost $7 a bushel while gain 2 costs $2 a bushel.

a) Formulate the objective function and inequality constraints. (5 marks)

b) Graph the inequality functions and shade the feasible region. (10 marks)

c) Determine the least cost combination of gains that will fulfill all the nutritional requirements. (5 marks)

d) What is the minimum cost that the manufacturer can incur? (5 marks)

Solution.

Summary of the above information

	Carbohydrate	Protein	Fructose	Cost
Grain y_1	24	14	8	$7
Grain y_2	4	7	32	$2
Total	128	168	120	

a) Objective function

Costs = $7y_1 + 2y_2$

Inequality constraints

Carbohydrate: $24y_1 + 4y_2 \geq 128$

Protein : $14y_1 + 7y_2 \geq 168$

Fructose: $8y_1 + 32y_2 \geq 120$

Non-negativity constraints/ condition

$y_1 \geq 0,\ y_2 \geq 0$

b) Graphying the inequality, we change the inequality into equations and we obtain the coordinates as follows

Carbohydrate

$24y_1 + 4y_2 = 128,$

y_1	0	5.3
y_2	32	0

Protein

$14y_1 + 7y_2 = 168$

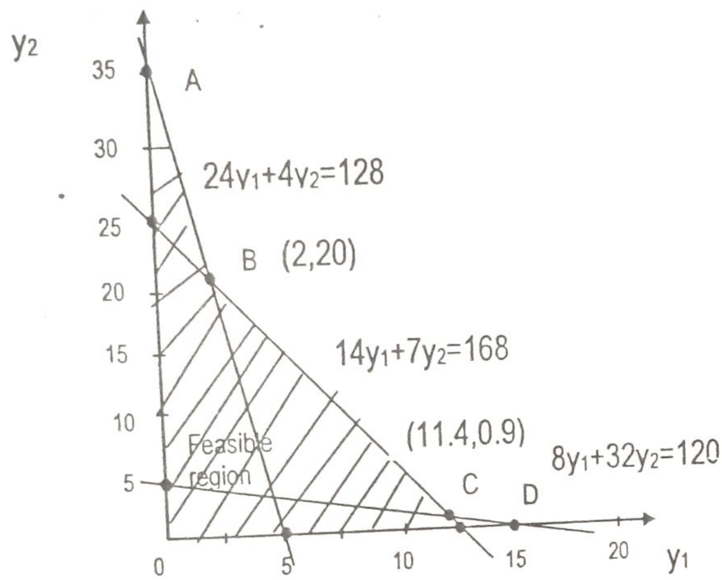
y_1	0	12
y_2	24	0

Fructose

$8y_1 + 32y_2 = 120$

y_1	0	15
y_2	3.8	0

Illustration



c. Determine the least cost.

Pair 1

$24y_1 + 4y_2=128$......... (i)

$14y_1 + 7y_2=168$......... (i)

Pair 2

$8y_1 + 32y_2=120$......... (i)

$14y_1 + 7y_2=168$......... (i)

Solving the pairs of equation simultaneously to obtain the minimum least cost.

Pair 1

$24y_1 + 4y_2 = 128$......... (i)

$14y_1 + 7y_2 = 168$......... (i)

Divide eq (i) by 4 and eq (ii) by 7 gives

$6y_1 + y_2 = 32$

$-2y_1 + y_2 = 24$

$4y_1 + 0 = 8$

$$\frac{4y_1}{4} = \frac{8}{4}$$

$y_1 = 2$

Substituting the values of y1 = 2 in either eq (i) or (ii) gives

$24(2) + 4y_2 = 128$

$48 + 4y_2 = 128$

$4y_2 = 128-48$

$$\frac{4y_2}{4} = \frac{80}{4}$$

$y_2 = 20$

Pair 2

$8y_1 + 32y_2 = 120$......... (i)

$14y_1 + 7y_2 = 168$......... (i)

Divide eq (i) by 8 and eq (ii) by 14 gives

$y_1 + 4y_2 = 15$

$y_1 + \frac{1}{2}y_2 = 12$

$0 + 3.5y_2 = 3$

$$\frac{3.5y_2}{3.5} = \frac{3}{3.5}$$

$y_2 = 0.9$

Substituting the values of y_2 in any equation gives.

$8y_1 + 32(0.9) = 120$

$8y_1 + 28.8 = 120$

$8y_1 = 120 - 28.8$

$$\frac{8y_1}{8} = \frac{91.2}{8}$$

$y_1 = 11.4$

To determine the least cost combination we substitute each pair in the objective function the one which will give the minimum cost it will be the considered requirements.

Objective function

Cost = $7y_1 + 2y_2$

B(2,20) Cost = 7(2) + 2(20)

Cost = 14 + 40

Cost = $54

C(11.4,0.9) Cost = 7(11.4) + 2(0.9)

Cost = 79.8 + 1.8

Cost = $81.6

Therefore the least cost combination of gains that will fulfill all the nutritional requirements are grain y_1,2units and grain y_2, 20 unit to minimize the cost of $54.

d) The minimum cost

$7y_1 + 2y_2$ = Cost

Where cost = $54

$7y_1 + 2y_2 = 54$

$$\frac{2y_2}{2} = \frac{54}{2} - \frac{7y_1}{2}$$

$$y_2 = 27 - \frac{7y_1}{2}$$

It shows the minimum cost of $54.

Infeasibility

This occurs in the case where there is no solution that satisfies the constraints. When graphed, infeasibility shows that there is no area of feasible solutions which satisfy all the constraints. Infeasibility implies that more resources may be required to satisfy the objective.

Question 1

Given the objective and constraints as follow;

$P = 2x + 10y$ (maximize)

Constraints

$2x + y \leq 6$

$5x + 4y \geq 40$

Show that they are infeasible.(4 marks)

Solution

$2x + y \leq 6$, $5x + 4y \geq 40$

Change the inequalities in the equations

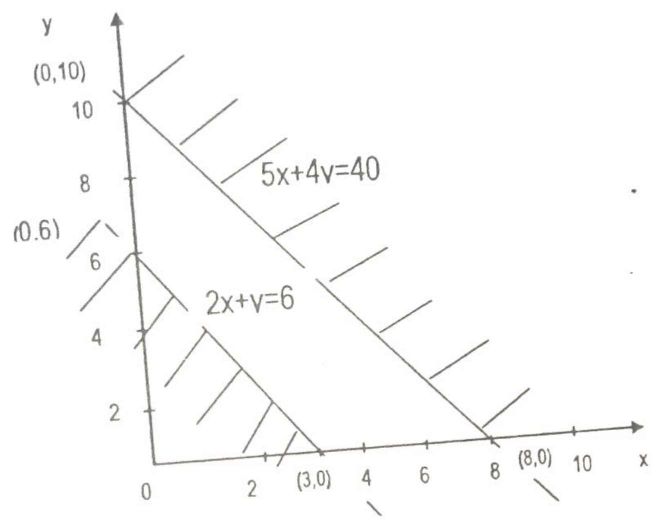
$2x + y = 6$, $5x + 4y = 40$

Obtained the coordinates

x	0	3
y	6	0

x	0	8
y	10	0

Illustration



The two equations are infeasible because they do not merge, hence no optimum solution can be found.

Unboundedness

This shows that there are some sets of constraints for which the region of feasible solution is not bounded. This implies incorrect formulation of the problem.

Question 1

Given the constraints

$2x + y \geq 6$

$5x + 4y \geq 40$

Show that they are unbounded.(3 marks)

Solution

Change all the inequalities into equations

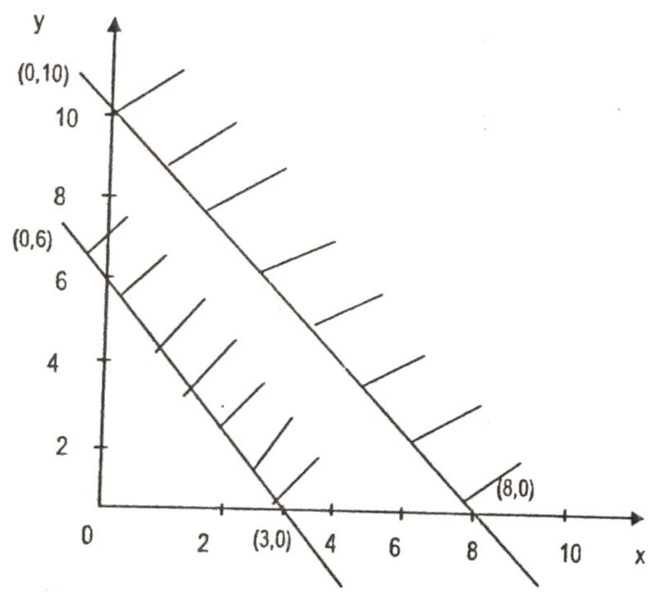
$2x + y = 6,$

$5x + 4y = 40$

X	0	3
Y	6	0

x	0	8
y	10	0

Illustration



Simplex method

Simplex method is another linear programming mathematical technique that is used to maximize or minimize a quantity by choosing appropriate values for variables involved.

Therefore simplex method can be used to solve both equations having only two variables and those with three and more variables.

Features of simplex method

Converting inequalities into equality equation.

Adding slack variables in each of the inequality equation. Therefore slack variable in this case represents the spare capacity or unused units.

Steps followed in the simplex method

Obtain the linear programming model.

Each constraint should be converted into an equation by adding a slack variable.

Formulate an augmented matrix and use it to set up initial tableau.

Select the pivot column in the initial tableau. This is the column with the highest contribution in the objective function row.

Select the pivot row. This is the row with the smallest non-negative quotient.

Calculate the new values for pivot raw by dividing every number in the row by the pivot number.

Replace the slack variable in the pivot row with the basic variable.

Question 1

A firm produces two types of products as being summarized in the table below.

	Carpentry	Finishing	Total profit
Number of product			
Tables (x)	4	2	70
Chairs (y)	3	1	50
Hours available	240	100	

Determine the number of chairs and tables to be produced in a week in order to maximize profit using simplex method. (15 marks)

Solution

Step 1: Rearranging the data into inequalities

Constraints

Carpentry: $4x + 3y \leq 240$

$2x + y \leq 100$

Non-negativity $x \geq 0, y \geq 0$

Objective function: profit (P) = $70x + 50$

Step 2: Convert inequalities into equations by adding slack variables

$4x + 3y + 1a + 0b = 240$

$2x + y + 0a + 1b = 100$

The slack variables can be included in the objective function with zero coefficients.

$P = 70x + 50y + 0a + 0b$

Step 3: We write the equations into matrix form of a3 x 4 augmented matrix

$$\begin{pmatrix} 4 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 70 & 50 & 0 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ a \\ b \end{bmatrix} = \begin{bmatrix} 240 \\ 100 \\ 0 \end{bmatrix} \text{ or } \begin{pmatrix} 4 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 70 & 50 & 0 & 0 \end{pmatrix} = \begin{bmatrix} 240 \\ 100 \\ 0 \end{bmatrix}$$

In the simplex method, the augmented matrix is referred to as the "tableau".

Initial tableau is;

	Solution variable	x	Y	A	B	Solution quantity
R_1	A	4	3	1	0	240
R_2	B	2	1	0	1	100
R_3		70	50	0	0	0

Step 4: Consider the highest contribution in the objective function row which forms the pivot column which is 70

Step 5: Consider the pivot row is row 2 since 100 is the smallest.

Step 6: Calculate the new values for the pivot row

	Solution variable	X	y	A	B	Solution quantity
R_1	A	4	3	1	0	240
$R_4 = R2/_2$	B	1	$\frac{1}{2}$	0	$\frac{1}{2}$	50
R_3		70	50	0	0	0

Step 7: Replace the slack variable b (in the pivot row) with the basic variable x in the pivot column

	Solution variable	X	Y	a	B	Solution quantity
$R_5 = (R_1 - 4R_4)$	A	0	1	1	-2	40
R_4	X	1	$\frac{1}{2}$	0	$\frac{1}{2}$	50
$R_6 = (R_3 - 70R_4)$		0	15	0	-35	-3,500

Other steps

Set up a new tableau by selecting the next pivot column y should replace slack variable a in the solution variable column

	Solution variable	x	Y	A	B	Solution quantity
R_5	A	0	1	1	-2	40
R_4	X	1	$\frac{1}{2}$	0	$\frac{1}{2}$	50
R_6		0	15	0	-35	-3,500

Calculate new values for the pivot row, as the pivot number is already 1

Use row operation to make all numbers in the pivot column equal to zero, accept for the pivot number.

	Solution variable	X	Y	A	B	Solution quantity
R_5	Y	0	1	1	-2	40
$R_7^{\cdot} = \left(R_4 - \frac{1}{2}R_5\right)$	X	1	0	$\frac{-1}{2}$	$\frac{3}{2}$	30
$R_8 = (R_6 - 15R_5)$		0	0	-15	-5	-4,100

Since there are no positive contributions in the objective function row the optimum solution has been reached.

Therefore maximum profit of Shs.4,100 occurs when 30 tables and 40 chairs are made. There are no unused hours.

The values of row 8 for slack variables are a great importance. These are the valuations of resources and are known as shadow prices.

a = -15, means that for every extra carpentry labour hours available, Shs.15 extra overall contribution would be gained.

b = -5, means that for every extra finishing labour hour availed; the overall contribution would increase by Shs.5.

Question 2

DW-Company makes doors and windows. Each door requires 1 hour on machine A and 2 hours on machine B. Each window requires 3 hours on machine A and 1 hour on machine B. The company has a maximum of 9 hours available on machine A and a maximum of 8 hours available on machine B. The company can realize a profit of $10 dollars on each door and $20 dollars on each window, letting x_1 and x_2 to represent the number of doors and the number of windows respectively.

Required

i) Formulate a linear programming model for the above information. (4 marks)

ii) Use simplex tableau method to determine the units of doors and windows that should be made in order to maximize profit. (7 marks)

iii) State the maximum profit. (1 mark)

Solution

	Machine A	Machine B	Profits
Door (X_1)	1	2	10
Window (X_2)	3	1	20
Total hours	9	8	

i) Linear programming model

Objective function

Profit (P) = $10x_1 + 20x_2$

Constraints

$X_1 + 3X_2 \leq 9$

$2X_1 + X_2 \leq 8$

Non-negativity

$X_1 \geq 0, X_2 \geq 0$

Converting inequality into equations and we add the slacks.

$X_1 + 3X_2 + 1a + 0b = 9$

$2X_1 + X_2 + 0a + 1b = 8$

$10X_1 + 20X_2 + 0a + 0b$

ii) Following the above steps

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 10 & 20 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 0 \end{pmatrix}$$

Initial tableau

	X_1	X_2	A	B	Solution quantity
R_1	1	3	1	0	9
R_2	2	1	0	1	8
R_3	10	20	0	0	0

	X_1	X_2	A	B	Solution quantity
$R_4 = (R_1 - 3R_2)$	-5	0	1	-3	-15
R_2	2	1	0	1	8
$R_5 = (R_3 - 20R_2)$	-30	0	0	-20	-160

	Solution variable	X_1	X_2	A	B	Solution quantity
$R_6 = \left({}^{R_4}/_{-5}\right)$	X_1	1	0	$\frac{-1}{5}$	$\frac{3}{5}$	3
$R_7 = (R_2 - 2R_6)$	X_2	0	1	$\frac{2}{5}$	$\frac{-1}{5}$	2
$R_8 = (R_5 - 30R_6)$		0	0	-6	-2	-70

Since there are no positive contributions in the objective function row. Therefore the optimum solution has been reached.

ii) Therefore the required units of doors (x_1) and windows (x_2) that should be made in order to maximize profits are 3 and 2 respectively.

iii) The maximum profit is $70 which is being maximized with respects to the required units.