DIFFERENTIAL CALCULUS

Basics of Differential Calculus

Calculus is an Italian word meaning **"small stone".** In its basic form calculus is like understanding something by looking at its **small constituent pieces**.

In mathematics, calculus looks at rates of change and what is accumulated when change takes place. This brings up two forms of calculus – **Differential and integral calculus.**

Definition: Differential calculus is the study of rate of change of a variable with respect to changes in other variables on which it depends.

Examples: R = **q**p

1. A change of distance covered per unit time
2. A change of demand for a unit as a result of unit change in the price
3. A change of population as a result of each additional year

Consider a variable x have a first value x_1 and a second value x_2. The change in value of x is

$x_2 - x_1$ and is termed the Increment in x and denoted as **Δx**. The Greek letter Δ (delta) has become a standard notation for increment in any variable.

Given a function f with a value of y when on x, we can write that y= f(x) and with specific values of $x_1$and x_2, we can say that,

$y_1 = f(x_1)$ and $y_2 = f(x_2)$

Thus, $\Delta y = f(x_2) - f(x_1)$

Now if write that $x_2 – x_1 = \Delta x$, then we can say that $x_2 = x_1 + \Delta x \Rightarrow \Delta y = f(x_1 + \Delta x) - f(x_1)$

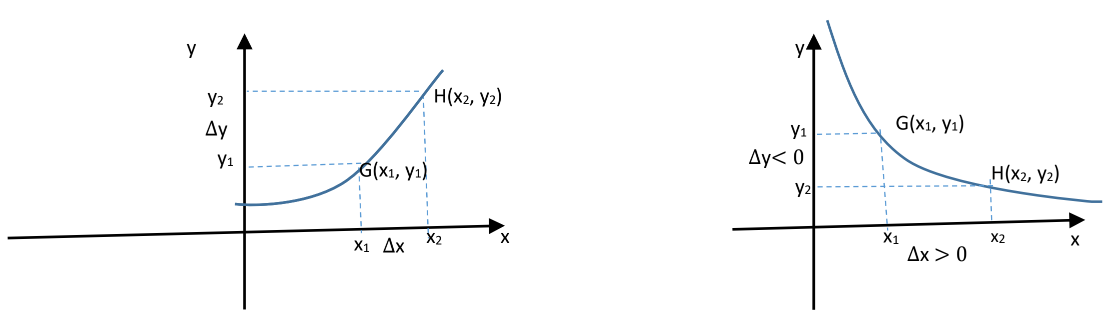
Since x_1 can be any value, we can write generally that,

$$\mathbf{\Delta y = f(x + \Delta x) - f(x)}$$

Since f(x) = y, we can write that,

$$\mathbf{\Delta y = f(x + \Delta x) – y \Rightarrow y + \Delta y = f(x + \Delta x)}$$

Let us examine this graphically,



(a) Figure 1 (b)

We see from figure (a) that Δx is the horizontal distance from G to H while Δy is the vertical distance from G to H. It should be noted the same is true with figure (b). However, the difference is that in (a), both Δx and Δy are positive while in (b) Δx $>$0 while Δy $<$ 0.

Example:

Consider a function $f(x) = 2x^2$. We noted that if y = f(x) then

Δy = f(x +Δx) - f(x). Let us look at a specific point where x =2 and the small change in x, Δx = 0.1. What is the value of Δy.

From above expression,

Δy = f(x +Δx) - f(x)

$= 2(2 + 0.1)^2 – 2(2)^2$

= 2(4.41) – 8

= 8.82 – 8 = 0.82.

Let us examine this graphically

We note that Δy = f(x +Δx) - f(x) is a change in y for a change Δx in x.

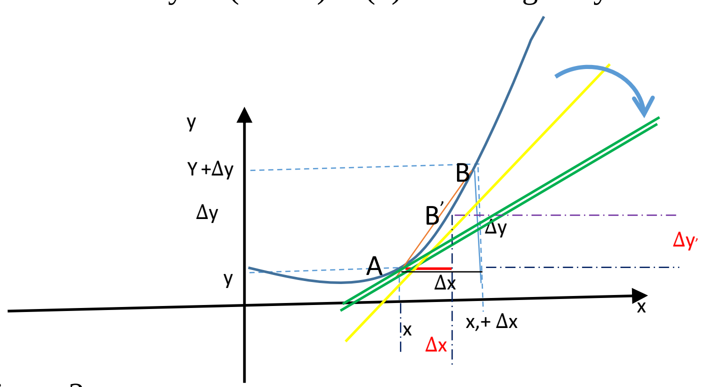


Figure 2

We see from figure 2 that the average rate of change of y over the change Δx in x is

$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

From the graph, this is the gradient of the line AB. As the line rotates clockwise, through yellow to green, we see that the new line AB/ (yellow line) indicating the point of curve intercept is moving towards the fixed point A and Δx/ and Δy/ becoming smaller.

The green line finally just touches the curve at point A and we say I is tangential to the curve. At this point Δx/ and Δy/ are infinitesimally small almost tending to zero and $\frac{\Delta y'}{\Delta x'}$ constitutes the gradient of the green line (the tangent to the curve at A) which is the gradient of the curve as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} =$$

The derivative of a function f(x) is defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f^{/}(x) = \frac{dy}{dx}$$

This may also be expressed as

$\frac{d(y)}{dx}$, $\frac{df}{dx}$ or $y^{/}$ or $D_x y$ or $D_x f$.

The process of finding the derivative of a function is differentiation of the function f.

Example

Given y= x^2 differentiate y with respect of x.

Solution

Let y = f(x) = x^2

Consider a small change in x, Δx

We have $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$

$= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$

Therefore,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) \Rightarrow \frac{dy}{dx} = 2x$$

Example

Find the derivative of f(x) = $2x^2 + 3x + 1$

Solution

$$\frac{\Delta y}{\Delta x} = \frac{2(x+\Delta x)^2 + 3(x+\Delta x) + 1 - (2x^2 + 3x + 1)}{\Delta x} = \frac{2x^2 + 4x\Delta x + 2(\Delta x)^2 + 3x + 3\Delta x + 1 - (2x^2 + 3x + 1)}{\Delta x}$$

$$= \frac{4x\Delta x + 2(\Delta x)^2 + 3\Delta x)}{\Delta x} = 4x + 2\Delta x + 3$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4x + 3 + 2\Delta x) \Rightarrow \frac{dy}{dx} = 4x + 3.$$

Example

Consider a general function,

$Ax^3 + Bx^2 + Cx + D$, and find $\frac{dy}{dx}$.

$y + \Delta y = A(x + \Delta x)^3 + B(x + \Delta x)^2 + C(x + \Delta x) + D$

$= A[x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3] + B[x^2 + 2x\Delta x + (\Delta x)^2] + Cx + C\Delta x + D$

$$=> \frac{\Delta y}{\Delta x} = \frac{\{A[x^3 + 3x^2\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3] + B[\,x^2 + 2x\Delta x\ + (\Delta x)^2] + Cx + C\Delta x + D\} - \{\,Ax^{\,2} + Bx^2 + Cx + D}{\Delta x}$$

$$=> \frac{\Delta y}{\Delta x} = \frac{\{A[x^3 + 3x^2\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3] + B[\,x2 + 2x\Delta x\ + (\Delta x)^2] + Cx + C\Delta x + D\} - \{\,Ax^3 + Bx^2 + Cx + D}{\Delta x}$$

$$=> \frac{\Delta y}{\Delta x} = \frac{3x^2\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3 + 2Bx\Delta x\ + B(\Delta x)^2 + C\Delta x}{\Delta x} = 3Ax^2 + 3Ax\,(\Delta x) + A(\Delta x)^2 + 2Bx + B(\Delta x) + C$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (3Ax^2 + 3Ax\,(\Delta x) + A(\Delta x)^2 + 2Bx + B(\Delta x) + C) => \frac{dy}{dx} = 3Ax^2 + 2Bx + C.$$

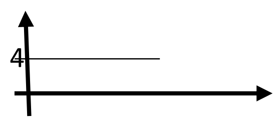
In the example above,

$f(x) = 2x^2 + 3x + 1$

we see that A = 0, B = 2, C= 3 and D= 1,

Filling in the above general formula, we have

$\frac{dy}{dx} = 2(2)x + 3 = 4x + 3.$



Example

y= f(x) = 4

Using the above general formula, we have A =B =C= 0 and D=4

$\frac{dy}{dx} = 3Ax^2 + 2Bx + C$ => that $\frac{dy}{dx} = 0 + 0 + 0 = 0$

Therefore the derivative of a constant function is 0.

Consider

y= f(x) = x

In this case A=B=0, C=1 D=0

Hence $\frac{dy}{dx} = 3Ax^2 + 2Bx + C = 0 + 0 + 1 = 1$

Consider $y = x^2$

A=0 B=1 C=0 and D=0

Hence $\frac{dy}{dx} = 3Ax^2 + 2Bx + C = 0 + 2(1)x + 0 = 2x$

From the above we can conclude that

$y = f(x) = x^n => \frac{dy}{dx} = f^{/}(x) = nx^{n-1}$

This is referred to as the **Power formula** of differentiation.

Examples

$\frac{d(x^3)}{dx} = 3x^{2-1} = 3x^2$

$\frac{d(x^{1/2})}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{d(x)}{dx} = 1x^{1-1} = x^0 = 1$

Rules for Differentiation

1. Derivative of a product of a constant and a function
   $\frac{d(cu)}{dx} = c\,\frac{d(u)}{dx}$ where u = f(x).
   For example, $\frac{d(cx^n)}{dx} = c\frac{d(x^n)}{dx} = cn\, x^{n-1}$.
2. Derivative of the sum of two functions u and v
   $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

   For example $y = x^3 + x^{1/3}$

   $\frac{dy}{dx} = 3x^2 + \frac{1}{3}x^{-2/3}$

Other examples

1. Differentiate the following:-

i) $\sqrt{x} - \frac{1}{x^2}$

$\frac{d(\sqrt{x} - \frac{1}{x^2})}{dx} = \frac{d(\sqrt{x})}{dx} - \frac{d(\frac{1}{x^2})}{dx} = \frac{d(x)^{1/2}}{dx} - \frac{d(x)^{-2}}{dx} = \frac{1}{2}x^{-1/2} + 2x^{-3}$

$= \frac{1}{2\sqrt{x}} + \frac{2}{x^3}$

ii) $\frac{v^2+3v-2}{v^2}$

$\frac{d\left(\frac{v^2+3v-2}{v^2}\right)}{dv} = \frac{d(1)}{dv} + \frac{d\left(\frac{3}{v}\right)}{dv} - \frac{d\left(\frac{2}{v^2}\right)}{dv} = 0 - 3v^{-2} + 4v^{-3} = \frac{4}{v^3} - \frac{3}{v^2}$ (note $1 = v^0$)

iii) The population(in thousands) of a city, at time t (years) is given by
$P(t) = 20,000 + 500t - 50t^2$
Find the instantaneous growth rate after 3 years.
$P^/(t) = \frac{d(20,000 + 500t - 50t^2)}{dt} = 0 + 500 - 100t$
At t =3, $P^/(3) = 500 - 100(3) = 200$.
The growth rate at 3 years is 200,000 per year.

Derivatives of Product and Quotients of functions

Product Rule

Given functions v(x) and u(x), then

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \text{ or } (uv)' = uv' + vu'$$

Proof

Let y = uv, then

$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\Delta v + v\Delta u + \Delta u\Delta v$

Subtracting y = uv from both sides we have

$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v$

Dividing both sides by Δx, we have,

$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u\Delta v}{\Delta x}$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = u\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} + v\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\Delta u\left(\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\right)$$

As Δu →0 as Δx →0, hence in the limit we have

$$\boldsymbol{\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}},$$

which is referred to as the product rule also represented as

$$\mathbf{(uv)' = uv' + vu'}$$

Example 1

Given that $y = (x + 1)(2x^3 + 3)$, evaluate $\frac{dy}{dx}$.

Let u = x + 1 and $v = 2x^3 + 3$

According to the product rule,

$\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, thus

$\frac{dy}{dx} = (x + 1)\frac{d(2x^3 + 3)}{dx} + (2x^3 + 3)\frac{d(x + 1)}{dx}$

$= (x + 1)(6x^2) + (2x^3 + 3)(1) = 6x^3 + 6x^2 + 2x^3 + 3 = 8x^3 + 6x^2 + 3$

Therefore,

$\frac{dy}{dx} = 8x^3 + 6x^2 + 3$

Example 2

Given that $y = (t^3 + 1)(t - \frac{1}{t^2})$, evaluate $\frac{dy}{dt}$.

Solution

Let $u = t^3 + 1$and $v = t - \frac{1}{t^2}$

According to the product rule,

$\frac{dy}{dt} = \frac{d(uv)}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}$, thus

$\frac{dy}{dt} = (t^3 + 1)\frac{d(t - \frac{1}{t^2})}{dt} + (t - \frac{1}{t^2})\frac{d(t^3 + 1)}{dt}$

$= (t^3 + 1)(1 + \frac{2}{t^3}) + (t - \frac{1}{t^2})(3t^2)$

$= t^3 + 2 + 1 + \frac{2}{t^3} + 3t^3 - 3$

$= 4t^3 + \frac{2}{t^3}$.

Example 3

Analysis of a countries economic data revealed that the average income per capita at time t is given by $\mathbf{W(t) = 6,000 + 500t + 10t^2}$, where **W** is in shillings and t is number of years from a base year. It was also established that the population size, in millions, is given by $\mathbf{P(t) = 10 + 0.2t + 0.01t^2}$**.** Use this information to find the rate of growth of GNP at time t.

Solution

Let GNP as a function of time be G(t), we can say that

$$G(t) = \text{Population x Per capita income} = P(t) \times W(t)$$

Thus $G(t) = (10 + 0.2t + 0.01t^2)(6000 + 500t + 10t^2)$

If we consider $v = 6000 + 500t + 10t^2$ and $u = 10 + 0.2t + 0.01t^2$, we can say that

$G(t) = uv$.

We know that

$G'(t) = \frac{d(uv)}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}$ ……………………………………

Therefore $G^{/}(t) = [(10 + 0.2t + 0.01t^2)(500 + 20t)] + [(6000 + 500t + 10t^2)(0.2 + 0.02t)]$

$= [5000 + 300t + 9t^2 + 0.2\,t^3] + [1200 + 220t + 12t^2 + 0.2t^3]$

$= 6{,}200 + 520t + 21t^2 + 0.4t^3$

Thus the rate of change of GNP, that is $G^{/}(t)$ is given by the expression

$6{,}200 + 520t + 21t^2 + 0.4t^3$.

Example

Example

Using the product rule of differentiation find $\frac{du}{dx}$ if $u=\left(x+\frac{3}{x}\right)^3 x$

Use of rule for differentiation of a product of functions.

$\frac{d(vu)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$, hence $\frac{du}{dx} = \left(x+\frac{3}{x}\right)^3 + 3x\left(x+\frac{3}{x}\right)^2.\,(1-\frac{3}{2x^2})$ **(2 Marks)**

$= \left(x+\frac{3}{x}\right)^2\left[\left(x+\frac{3}{x}\right) + 3(\frac{2x^2-3}{2x})\right] = \left(x+\frac{3}{x}\right)^2\left[\left(x+\frac{3}{x}\right) + 3(\frac{2x^2-3}{2x})\right]$ **(1 Mark)**

$=\left(y+\frac{3}{y}\right)^2\left[(\frac{8y^2-3}{2y})\right]$ **(2 Marks)**

Theorem 2 Quotient Rule

Given functions v(x) and u(x), then, we can say let $y = \frac{u}{v}$.

$$\frac{dy}{dx} = \frac{d(\frac{u}{v})}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \text{ or } \left(\frac{u}{v}\right)^{/} = \frac{vu^{/} - uv^{/}}{v^2}$$

Proof

Let $y = \frac{u}{v}$, then

$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$

$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v(u + \Delta u) - u(v + \Delta v}{v(v + \Delta v)} = \frac{v(\Delta u) - u(\Delta v)}{v(v + \Delta v)}$

Dividing both sides by Δx, we have,

$\frac{\Delta y}{\Delta x} = \frac{v(\Delta u) - u(\Delta v)}{\Delta x[v(v + \Delta v)]}$

As Δx →0, Δu →0, and Δv →0,

So we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{v \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) - u \lim_{\Delta x \to 0}\left(\frac{\Delta v}{\Delta x}\right)}{v(v + \Delta v)}$$

$\Rightarrow \boldsymbol{\frac{dy}{dx}} = \boldsymbol{\frac{d(\frac{u}{v})}{dx}} = \boldsymbol{\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$ which can also be written as $\boldsymbol{\left(\frac{u}{v}\right)^{/} = \frac{vu^{/} - uv^{/}}{v^2}}$

Example

Use the Quotient Rule of Differentiation

$$\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Let u = e^{2x} then $\frac{du}{dx} = 2e^{2x}$

and v = $2x^2 + 5$ then $\frac{dv}{dx}$ =4x

hence $f^{/}(x) = \frac{(2(2x^2+5)e^{2x} - 4xe^{2x}}{(2x^2+5)^2} = \frac{2e^{2x}(2x^2 - 2x + 5)}{(2x^2+5)^2}$

Chain Rule

If y = f(u) and u = g(x), then

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Proof

y + Δy = f(u + Δu) and u + Δu = g(x + Δx)

So we can say that

Δy = f(u + Δu) – f(u) …………………………………………………1

Δu = g(x + Δx) – g(x)……………………………………………………….(ii)

$\frac{\Delta y}{\Delta u} = \frac{f(u + \Delta u) - f(u)}{\Delta u}$…………………………………………………………………(iii)

$\frac{\Delta u}{\Delta x} = \frac{g(x + \Delta x) - g(x)}{\Delta x}$..(iv)

If we multiply equations (iii) and (iv), we get

$\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \left(\frac{f(u + \Delta u) - f(u)}{\Delta u}\right) x \left(\frac{g(x + \Delta x) - g(x)}{\Delta x}\right)$

$=> \frac{\Delta y}{\Delta x} = \left(\frac{f(u + \Delta u) - f(u)}{\Delta u}\right) x \left(\frac{g(x + \Delta x) - g(x)}{\Delta x}\right)$

As $\Delta u \rightarrow 0$ and $\Delta x \rightarrow 0$, then we have,

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example 1

$y = (2x + 8)^6$, find the expression for $\frac{dy}{dx}$.

Let y be composed of two functions f(u) and u(x), where y = (f ° u)(x).

$u(x) = 2x + 8$ and $f(u) = u^6$

Chain Rule says that

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u^5 (2) = 12u^5 = 12 (2x + 8)^5$

Example 2

$y = \left(t^2 + \frac{1}{t^2}\right)^5$, find the expression for $\frac{dy}{dt}$.

$y = u^5$ and $u = t^2 + \frac{1}{t^2}$

$\frac{dy}{du} = 5u^4$

$\frac{du}{dt} = 2t - \frac{2}{t^3}$

$=> \frac{dy}{dt} = 5\left(t^2 + \frac{1}{t^2}\right)^4 \left(2t - \frac{2}{t^3}\right) = 10\left[\left(t^2 + \frac{1}{t^2}\right)^4 \left(t - \frac{1}{t^3}\right)\right]$

Example 3

The manufacturer's cost function is given by,

$C(x) = 4{,}000 + 10x - 0.2x^2 + 0.003x^3$.

If the current production level is 200 and is increasing at the rate of 3 per month, determine the rate at which the production costs are increasing.

Solution

What is required is $\frac{dC}{dt}$.

We are given that $\frac{dx}{dt}$ = 3, but we know that $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$.

Since $C(x) = 4{,}000 + 10x - 0.2x^2 + 0.003x^3$, we have $\frac{dC}{dx} = 10 - 0.4x + 0.009x^2$,

Therefore $\frac{dC}{dt} = (10 - 0.4x + 0.009x^2)(3) = 30 - 1.2x + 0.027x^2$. Thus at x = 200

$\frac{dC}{dt} = 30 - 1.2(200) + 0.027(200^2) = 30 - 240 + 1080 = 870$

The rate of production costs are increasing by 870.

Example

A) A manufacturer's cost function is given by

$$C(x) = 2000 + 10x - 0.1x^2 + 0.002\,x^3.$$

If the current production level is x=100 and is increasing at a rate of 2 per month, find the rate at which the production costs are increasing.

Solution

The rate of change of cost with respect to x,

$\frac{dC}{dx} = C^{/}(x) = 10 - 0.2x + 0.006x^2$

At x= 100, $\frac{dC}{dx} = 10 - 20 + 60 = 50$

We are given that at x =100,

$\frac{dx}{dt} = 2$

By Chain Rule.

$\frac{dC}{dt} = \frac{dC}{dx} \times \frac{dx}{dt} = 50 \times 2 = 100$

Therefore production costs are increasing at a rate of 100.

Reciprocal Rule

If is differentiable at x and $f(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{f^{/}(x)}{[f(x)]^2}$$

If $u = f(x) \Rightarrow \frac{1}{f(x)} = \frac{1}{u}$, thus $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{d}{dx}\left[\frac{1}{u}\right] = \frac{1}{u^2}\frac{du}{dx} = \frac{f^{/}(x)}{[f(x)]^2}$ since $u = f(x)$.

Applications of the concept of Derivatives

Derivatives have practical applications in many areas, both in physical and economic aspects. They are useful in analysis of rates of change, marginal costs, revenues and yield as well as in determining the greatest and least values of functions.

Marginal Analysis

Under this topic we will look at Marginal cost and Revenue, marginal productivity and optimization of business parameters such as revenues, costs and profits. In this aspect, marginal value constitutes the average change per extra item, when a very small change is made in the amount of given level of activity.

Marginal cost

Definition: The limiting value of the average cost per extra item as the number of extra items approaches zero. So MC can be conceptualized as the average cost per extra item when a very small change is made in the amount produced.

Marginal Cost = $\lim\limits_{\Delta x \to 0} \frac{\Delta c}{\Delta x}$

For example, given that a cost function has been established to be

$C(x) = 400 + 0.02x^2$, if 200 items are produced per month, the cost will be

$C(200) = 400 + 0.02\,(200)^2 = 1{,}200$.

If the manufacturer is considering a change of production levels per month to $200 + \Delta x$. this represents an increment in cost , that is

$C + \Delta C = 400 + 0.02\,(200 + \Delta x)^2 = 400 + 0.02[40{,}000 + 400\Delta x + (\Delta x)^2]$

$= 400 + 800 + 8\Delta x + 0.02(\Delta x)^2$

$\Rightarrow \Delta C = [400 + 800 + 8\Delta x + 0.02(\Delta x)^2] - 1{,}200$

$= 8\Delta x + 0.02(\Delta x)^2$

=> The average cost per extra item is $\frac{\Delta c}{\Delta x} = 8 + \Delta x$

Thus, the Marginal cost = $\lim\limits_{\Delta x \to 0} \frac{\Delta c}{\Delta x} = 8$

In general, Marginal Cost = $\lim\limits_{\Delta x \to 0} \frac{\Delta c}{\Delta x} = \frac{dc}{dx}$

Thus if the cost function is $C(x) = 0.002x^3 - 0.4x^2 + 80x + 2{,}000$, then the marginal cost function is $C'(x) = \frac{dc}{dx} = 0.006x^2 - 0.8x + 80$.

While marginal cost function is $\frac{dc}{dx}$, the average cost is C(x) (total cost of units) divided by the number of units produced, that

Average cost per item = $\frac{C(x)}{x}$ presented by the symbol $\bar{C}(x)$.

Thus the marginal cost is the average cost per additional unit, while average cost is cost per unit on average.

Example

Given a cost function of

$C(x) = 2{,}000 + 5x + 0.2x^2$

Derive the marginal and average cost functions.

Solution

The marginal cost function

$C'(x) = \frac{dc}{dx} = 5 + 0.4x$

The average cost function $\bar{C}(x) = \frac{C(x)}{x} = \frac{2{,}000+5x+0.2x^2}{x} = \frac{2{,}000}{x} + 0.2x + 5$

Marginal Revenue and Profit

Just as in the case of Marginal cost, we define marginal revenue as additional revenue per additional unit when a small increment is made in the number of items sold and its function is $R'(x) = \frac{dR}{dx}$.

Example

Given that revenue function as

$R(x) = 20x - 0.02x^2$, where x is the number of items sold,

i) Determine the marginal revenue function
ii) What is the marginal revenue when x = 100?

Solution

i) The Marginal Revenue function is
$R'(x) = \frac{dR}{dx} = 20 - 0.04x$
ii) When 100 items are sold , the marginal revenue, is given by $R'(100) = 20 - 0.04(100)$
$20 - 4 = 16$. Thus when 100 items are sold. Any small increase in sales provides an increase in revenue of 16 per item.

Revenue and Marginal profit

Revenue is related to the price charged for each item sold. The law of supply and demand dictate that as the price increases the demand goes down. Also revenue is a product of products sold and the price charged. That is,

$$R(x) = xp$$

where p is the price per item and x is the number of items sold.

Example

Given that when x = 100, the demand equation is x = 2,000 – 200p. Determine the marginal revenue for the company when x = 200.

Solution

$R(x) = xp$,

but we know that $x = 2{,}000 - 200p \Rightarrow p = \frac{2{,}000 - x}{200}$

therefore,

$R(x) = x\left(\frac{2{,}000-x}{200}\right) = 10x - 0.005\,x^2$

The Marginal revenue function is

$R^{/}(x) = \frac{dR}{dx} = \frac{d(10x - 0.05x^2)}{dx} = 10 - 0.01x$

When x = 200,

$R^{/}(x) = 10 - 0.01(200) = 8$

Profit function

Profit function P(x) is given as

$P(x) = R(x) - C(x)$,

where R(x) and C(x) are revenue and cost functions respectively.

Therefore, the Marginal profit

$P^{/}(x) = R^{/}(x) - C^{/}(x)$

Example

Given the demand function as $p + 0.2x = 110$ and cost function as $c(x) = 2{,}000 + 10x$, determine the marginal profits when 100 units are sold and when 300 units are sold. Comment of the two results.

Solution

$R(x) = xp = x(110 - 0.2x) = 110x - 0.2x^2$

=> $P(x) = (110x - 0.2x^2) - (2,000 + 10x) = 100x - 0.2x^2 - 2,000$

Therefore, the Marginal Profit function, $P^{/}(x) = \frac{d(100x - 0.2x^2 - 2,000)}{dx} = 100 - 0.4x$

When x =100, MP = 100 -0.4(100) = 60, and when x = 300, MP = 100 – 0.4 (300) = -20.

When producing 100 units a small increase in production will yield an additional profit of 60 per unit while when producing at 300, and additional increment in the units will lead to a loss of 20 per additional unit.

Marginal productivity

Productivity is the amount of output from a company per given period. Consider a situation when a company has h as the number hours available per week and that this produces x amount of output. It can be said that

$x = f(h)$

If the amount of labour is given an increment of Δh then

$\Delta x = f(h + \Delta h) - f(h)$

The average increment with respect to h is

$\frac{\Delta x}{\Delta h} = \frac{f(h + \Delta h) - f(h)}{\Delta h}$

Thus the amount of average additional production per extra unit of labour corresponding to a given increase Δh is the Marginal productivity given as

$\frac{dx}{dh} = \lim_{\Delta h \to 0} \left(\frac{f(h + \Delta h) - f(h)}{\Delta h} \right)$

Thus marginal productivity of labour measures the increase in productivity per additional unit of labour, when a small change in the amount of labour is employed.

Marginal yield

Yield in financial terms means the amount earned from an investment. This is not to be confused with return on investment which is the total amount of profit generated divided by the amount invested.

If y is the yield function and s the investment, then,

Marginal Yield = $\frac{dy}{ds}$

Exercise

1. Calculate the marginal cost functions for the following functions:
   a) $C(x) = 200 + 2x – 0.06x^2 + 0.0002x^3$
   b) $C(x) = 10^{-4} x^3 – 0.02x^2 40x + 400$
   c) $C(x) = (\ln 4)x^3 + 50$
2. Given a revenue function,
   $R(x) = 100x – x^3(1 + \sqrt{x})$,
   find the marginal revenue function.
3. If the demand equation is $12p + 3x + 0.012x^2 = 720$,
   a) find the marginal revenue when p =15
   b) Find the value of x that makes $P^{/}(x) = 0$ and calculate the corresponding profit.
4. If the cost function is of the form, $Ax^2 + Bx + C$, at what value of x is the marginal cost equal to the average cost $\bar{C}(x)$.

Marginal Profit Example

A shoe manufacturer can use his plant to make either men's or women's shoes. If he makes x (in thousands of pairs) men's shoes and y (in thousands of pairs) women's shoes per week, then x and y are related by the equation,

$$2x^2 + y^2 = 25.$$

If the profit is \$10 on each pair of shoes, calculate the marginal profit with respect to x when x = \$8.

Derivatives of Exponential and logarithmic functions

An exponential function is one in which the independent (predictor) variable is at least an exponent of one of the terms. On the other hand, a logarithmic function is one in which, at least, one of the terms has the independent or dependent variable in logarithmic form.

Examples of exponential functions are

1. $y = 10^x$
2. $y = x + 2^{2x}$

Examples of logarithmic functions are

1. $y = \log_{10} 2x$
2. $\ln y = 3x + 2x^3$

Derivative of an exponential function

Consider a basic exponential function

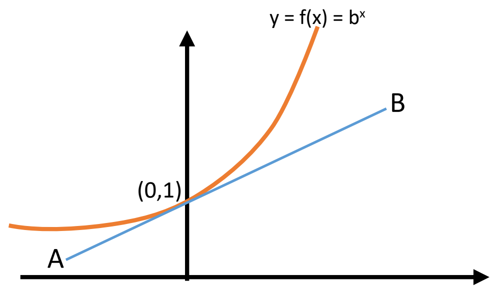
$y = f(x) = b^x$

This function has a number of special characteristics:

1. When x=0, y = 1 regardless of the value of b. This means that the y intercept is always (0,1).

2. The bigger the value of the **b** the greater the slope at (0,1).

Consider the graph below



Consider the gradient of the curve at x = 0, that is, (0,1) point. F

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(b^{\Delta x}) - b^0}{\Delta x}$$

We note that $b^0 = 1$ and let us consider a value of b such that $f'(0) = 1$, that is that gradient of AB and hence the slope of the curve at (0,1) is 1. Hence,

$$\lim_{\Delta x \to 0} \frac{(b^{\Delta x}) - 1}{\Delta x} = 1$$

The number which satisfies this equation is b = e denoted as e = 2.7183 **(Euler's Number after Leonhard Euler a Swiss Mathematician (1707-1783))** giving a function

$$\mathbf{y = e^x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(e^{x+\Delta x}) - e^x}{\Delta x}$$

$$e^{x+\Delta x} = e^x \cdot e^{\Delta x}$$

$$\frac{dy}{dx} = e^x \lim_{\Delta x \to 0} \frac{(e^{\Delta x}) - 1}{\Delta x}$$

We know for the number e, $\lim_{\Delta x \to 0} \frac{(e^{\Delta x}) - 1}{\Delta x} = 1$, therefore $\frac{dy}{dx} = e^x$.

Thus for the special number e, when,

$$\mathbf{y = e^x} \text{ then } \mathbf{\frac{dy}{dx} = e^x.}$$

This property makes natural exponential function very fundamental. Its derivative is always equal to the function itself.

Example

Evaluate $\frac{dy}{dx}$ if $y = x^2e^x$

Solution

Using the product rule, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

Now if $v = e^x$ and $u = x^2$, then

$$\frac{dy}{dx} = e^x(2x) + x^2(e^x) = e^x(x^2 + 2x)$$

Differentiation of logarithmic functions

Consider a logarithmic function $y = \ln x$. we can say that $x = e^y$.

$$\Rightarrow \frac{d(x)}{dx} = \frac{d(e^y)}{dx} \Rightarrow 1 = e^y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Hence if $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

Example

Find $\frac{dy}{dx}$ if $y = \ln(2x^2 + 3)$

Solution

Let $u = 2x^2 + 3 \Rightarrow y = \ln u$.

Therefore we can say that,

$\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 4x$

By Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 4x = \frac{4x}{2x^2+3}$$

In general, we can say,

$$\frac{d(\ln(\text{inside}))}{dx} = \frac{1}{\text{Inside}} \cdot \frac{d(\text{Inside})}{dx}$$

Example

Find $\frac{dy}{dx}$ if $y = \ln\frac{e^x}{\sqrt{x^2+1}}$

Solution

$y = \ln\frac{e^x}{\sqrt{x^3+1}}$ => $y = \ln e^x - \ln\sqrt{x^2+1}$

=> $y = x - \frac{1}{2}\ln(x^2+1)$

=> $\frac{dy}{dx} = 1 - \frac{1}{2}\left(\frac{1}{x^2+1} \cdot 2x\right) = 1 - \frac{x}{x^2+1}$

Example

B) Given that $y = x^2 \ln 2x$ find $\frac{dy}{dx}$

Solution

This is a case of Product of functions, hence the use of Product Rule

$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Let $v = x^2$ and $u = \ln 2x$

$\frac{dy}{dx} = \ln 2x(2x) + x^2 (\frac{2}{2x}) = 2x\ln 2x + x = x(2\ln 2x + 1)$

Example

Find $\frac{dy}{dx}$ if $y = \log_{10}(x^2+1)$

Let us consider $\log_a b = \frac{\log_c b}{\log_c a}$

If $x = \log_a b$ => $b = a^x$. Consider log base c for both sides, then $\log_c b = \log_c(a^x)$

=> $\log_c b = x \log_c a$ =>$x = \frac{\log_c b}{\log_c a}$, but $x = \log_a b$ => $\log_a b = \frac{\log_c b}{\log_c a}$.

Given this, then,

$y = \log_{10}(x^2+1) = \frac{\log_e(x^2+1)}{\log_e 10} = \frac{\ln(x^2+1)}{\ln 10}$

Hence $\frac{dy}{dx} = \frac{1}{\ln 10}\left[\frac{d(\ln(x^2+1))}{dx}\right] = \frac{2x}{\ln 10(x^2+1)}$

Summary of the Chain Rule

f(x)	$f^{/}(x)$
$(Inside)^n$	$n(Inside)^{n-1}$
$(u(x))^n$	$n(u(x))^{n-1}u^{/}(x)$
$(u(x))^n$	$n(u(x))^{n-1} \frac{d(u(x))}{dx}$
$e^{(inside)}$	$\frac{d(inside)}{dx}(e^{(inside)})$
$e^{(u(x))}$	$u^{/}(x)e^{(u(x))}$
$ln(inside)$	$\frac{1}{(inside)}\frac{d(inside)}{dx}$
$ln(u(x))$	$\frac{1}{(u(x))}\frac{d(u(x))}{dx}$

Example

(ii) Using your knowledge of logarithmic functions show that

$\ln\left(\frac{1}{\sqrt{x+2}}\right) = -\frac{1}{2}\ln(x+2)$

RHS = $\ln\left(\frac{1}{\sqrt{x+2}}\right) = \ln 1 - \ln(x+2)^{1/2} = 0 - \frac{1}{2}\ln(x+2) = -\frac{1}{2}\ln(x+2)$ = LHS

Therefore $\ln\left(\frac{1}{\sqrt{x+2}}\right) = -\frac{1}{2}\ln(x+2)$

From the above, $f^{/}(x) = \frac{d}{dx}\left(-\frac{1}{2}\ln(x+2)\right)$

Using chain rule method

Let u = x+ 2 $\Rightarrow \frac{du}{dx} = 1$ and $f^{/}(x) = -\frac{1}{2}\ \frac{d(lnu)}{du}.\ 1 = -\frac{1}{2u} = -\left(\frac{1}{2(x+2)}\right)$

Example

The growth function of a population is

$y = p_m(1 - ce^{-2kt})^3$ where p_m, c and k are constants. What is the growth rate at time t?

Solution

The answer is to find $\frac{dy}{dt}$.

$\frac{dy}{dt} = \frac{d\left[p_m\left(1 - ce^{-2kt}\right)^3\right]}{dt}$

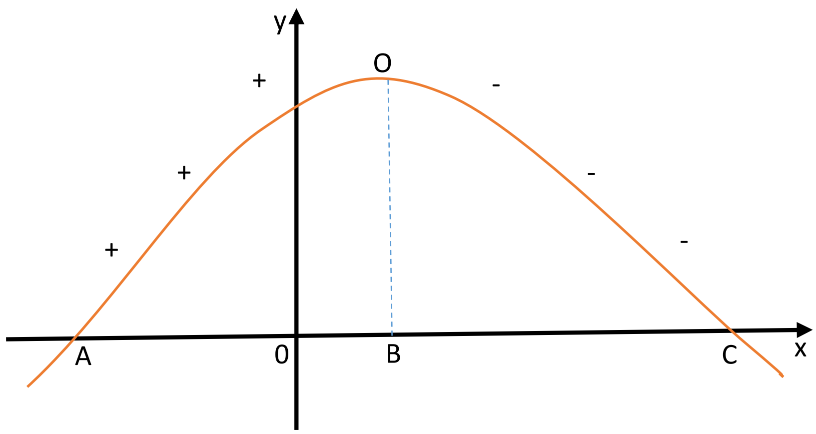
We know that $\frac{d(u(x))^n}{dx} = n\,(u(x))^{n-1}\ \frac{d(u(x))}{dx}$

If $u(t) = 1 - ce^{-2kt}$, then $\frac{d\left[p_m\left(1 - ce^{-2kt}\right)^3\right]}{dt} = p_m\,(3(u(t))^2)\,\frac{d\left[\left(1 - ce^{-2kt}\right)\right]}{dt}$

$\frac{dy}{dt} = 3p_m(1 - ce^{-2kt})^2[0 + 2cke^{-2kt}] = 6p_m cke^{-2kt}\,(1 - ce^{-2kt})^2$

Maxima and Minima or Optimization: The first derivative Test

Greatest and least values occur with varying values of the independent variable. Refer to the graph below:

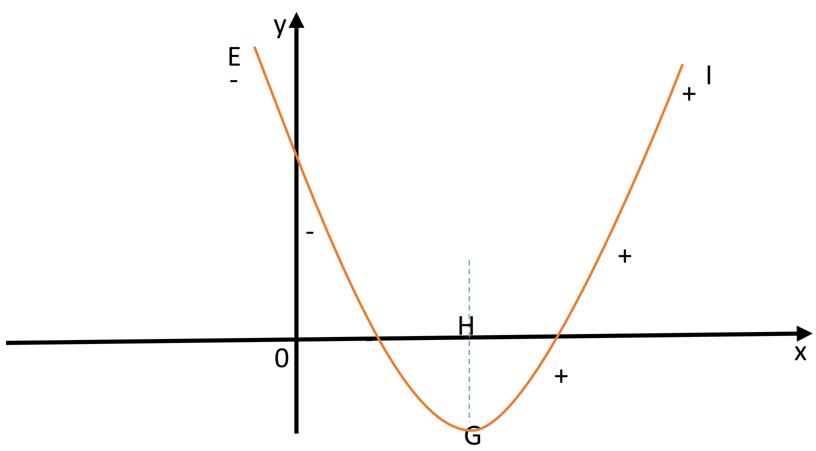


The figure shows a of parabola of the form,

$y = ax^2 + bx + c$

with $a < 0$ (that is negative). It is noted that between A and O, the gradient of the curve is positive and between O and C, the gradient is negative. We also note that at point O the gradient is changing from positive to negative. The value of y is maximum at O

When $a > 0$ then the parabola changes shape to the figure below



In this case $a > 0$ (that is positive). It is noted that between E and G, the gradient of the curve is negative and between G and I, the gradient is Positive. We also note that at point G the gradient is changing from negative to positive. The value of y is minimum at G.

From the above observations, we can say that at the maximum or minimum the slope of the curve is zero, which can be represented as

$\frac{dy}{dx} = f^{/}(x) = 0.$

Example

Find the greatest or least value of y on a curve

$y = 6x - x^2$

$\frac{dy}{dx} = 6 - 2x.$

The gradient is zero when

$6 - 2x = 0 \Rightarrow x = 3$

In order to establish whether this slope is at a minimum or maximum, we need to assess the changes of the value of $\frac{dy}{dx}$ in the vicinity of x = 3. Consider the graphical representation below:

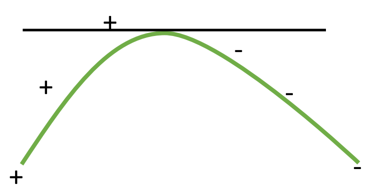


Fig. A Local Maximum

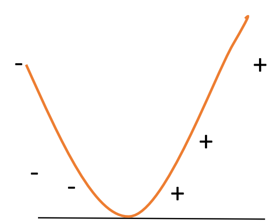


Fig B Local Minimum

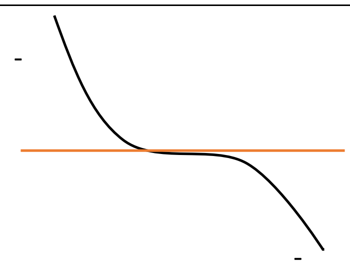


Fig. C Stationary point of Inflexion

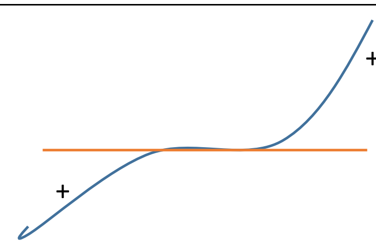
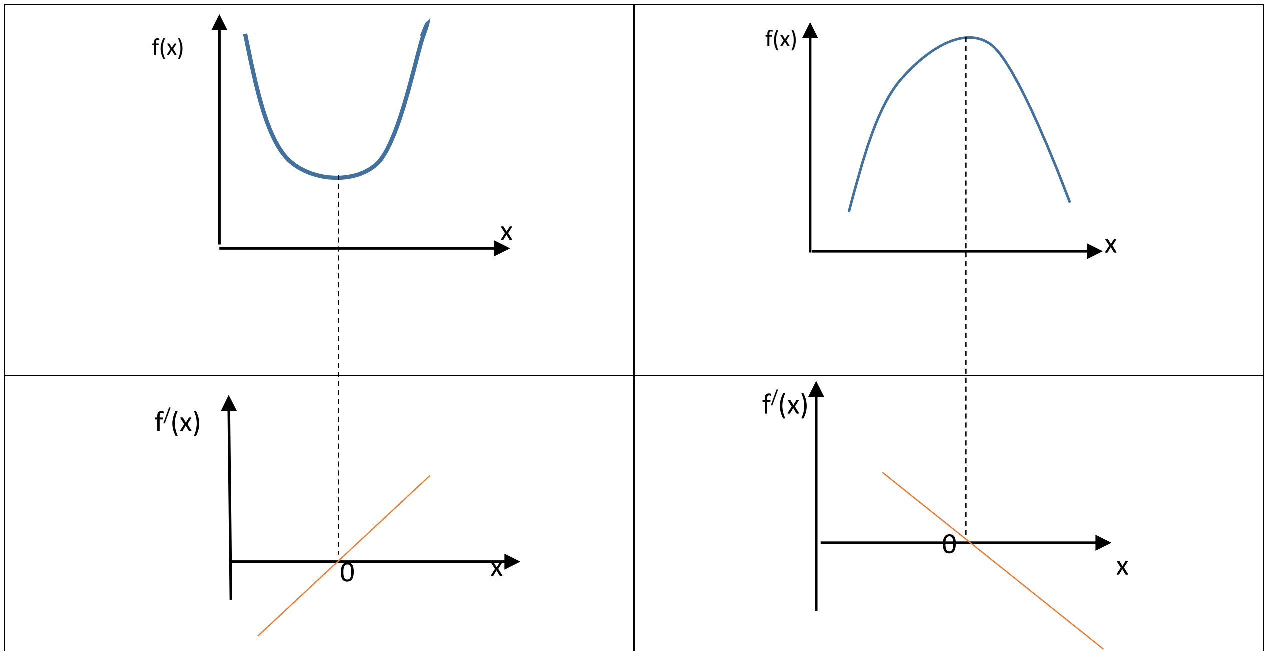


Fig. D Stationary point of Inflexion



Because $f^{//}(x) > 0$ – it has a minimum.	Because $f^{//}(x) < 0$ we say the graph –it has a maximum

Consider the curve in Fig A we note the value of $\frac{dy}{dx}$ to the left of the point where it is zero is positive and to the right it is negative. This shows the curve has a maximum value. On the other hand in Fig B we see that immediately to the left $\frac{dy}{dx}$ is negative and to right it is positive, meaning the curve has a minimum value when $\frac{dy}{dx} = 0$.

In the example above $\frac{dy}{dx} = 0$ when $x = 3$ and $y = 6(3) - (3)^2 = 9$ that at point (3,9).

When x is slightly less than 3, $\frac{dy}{dx} > 0$ ie positive and when x is slightly more than 3, $\frac{dy}{dx} < 0$ ie negative therefore the point (3,9) is the maximum value of y at y = 9.

We note that at any point where the gradient of the curve is zero, y is said to have a **STATIONARY Value**. These points constitute the **maximum or minimum point is called a turning point or critical points.**

Example 1

Given that $y = x^2 - 8x + 6$ find the value of x at maximum or minimum value of y.

$\frac{dy}{dx} = 2x - 8$ thus $\frac{dy}{dx} = 0 \Rightarrow 2x - 8 = 0 \Rightarrow x = 4$.

Value of x	L	4	R
Sign of $\frac{dy}{dx}$	Negative	0	Positive

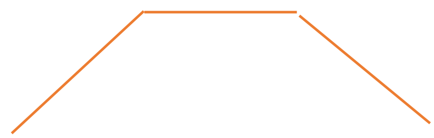
This means that y has a minimum value of y = $(4)^2 - 8(4) + 6 = -10$

Example

Given that y = 10 + 8x - x^2 find the value of x at maximum or minimum value of y.

$\frac{dy}{dx} = -2x + 8$ thus $\frac{dy}{dx} = 0 \Rightarrow -2x + 8 = 0 \Rightarrow x = 4$.

Value of x	L	4	R
Sign of $\frac{dy}{dx}$	Positive	0	Negative



This means that y has a maximum value of y = $10 + 8(4) - (4)^2 = 16$

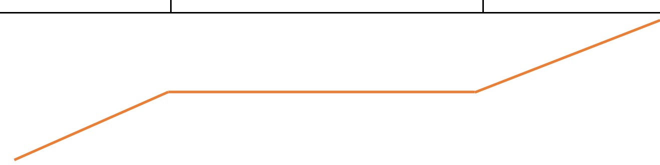
Recalling from the theory of functions, it noted that if $y = Ax^2 + Bx + C$. if $A > 0$ (positive) then we consider a minimum and if $A < 0$ (negative) then we are looking at a maximum.

Point of Inflexion

$Y = x^3 - 3x^2 + 3x$

$\frac{dy}{dx} = 3x^2 - 6x + 3$ thus $\frac{dy}{dx} = 0 \Rightarrow 3(x-1)^2 = 0 \Rightarrow x = 1$.

Value of x	L	1	R
Sign of $\frac{dy}{dx}$	Positive	0	Positive



This means that y has a minimum value of y = $(4)^2 - 8(4) + 6 = -10$

Application Examples

Example 1

A farmer has 600 m of fencing chain link wire mesh to enclose a rectangular field. He wants to utilize an existing wall as one of the sides. Express the area A of the field that can be fenced as a function of the length x of one side of it. What is the maximum area that can be fenced?

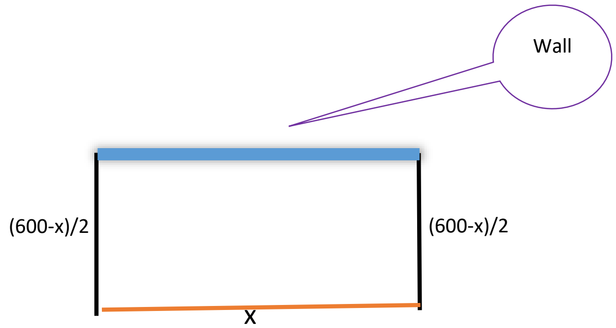


Figure: Fencing Dimensions

Area of a rectangle A = Length x Breadth.

Therefore

$A = x(600 - x)/2 = 300x - \frac{x^2}{2}$

Maximum area is when

$\frac{dA}{dx} = 0 \Rightarrow \frac{dA}{dx} = 300 - x = 0$

Hence A is maximum when x = 300 m

The maximum area that can be fenced $A_{max} = 300(300) - \frac{(300)^2}{2} = 90{,}000 - 45{,}000$

Therefore, $A_{max} = 45{,}000\ m^2$

Example 2

The demand per month, x, for a certain commodity at price p (in thousands) per unit is given by the relation

$$x = 1350 - 45p$$

The cost of labor and material to manufacture this commodity is 5(thousands) per unit and the fixed costs are 2,000 (thousands) per month. Derive the monthly profit as a function of quantity x sold and determine the quantity to be sold per month to realize maximum profit. What is the maximum profit?

Total cost = Fixed cost + Variable cost

$\Rightarrow C(x) = 2{,}000 + 5x$

Revenue R(x) = Price x Quantity sold = px.

We are given that $x = 1350 - 45p \Rightarrow p = \frac{1350 - x}{45}$

$=> R(x) = x \left(\frac{1350-x}{45}\right) = \frac{1350x-x^2}{45}$

Profit $P(x) = R(x) – C(x) = \frac{1350x-x^2}{45}$ - 2,000 + 5x = $\frac{1350x-x^2-90,000-225x}{45}$

=> The monthly profit function is $P(x) = \frac{1,125x-x^2-90,000}{45}$

P(x) is maximum when $\frac{dP}{dx} = 0 => \frac{d\left(\frac{1,125x-x^2-90,000}{45}\right)}{dx} = 0$ => 1,125 – 2x =0

Therefore, maximum profit will be realized when x = 562.5 units.

The maximum profit is $P(x) = \frac{1,125(562.5)-562.5^2-90,000}{45} =$

$= \frac{632,812.5-316,406.25-90,000}{45} = 5,031.25$

That is P_{max} = 5,031,250

Example 3

The cost of producing x items per week is

$C(x) = 1000 + 6x – 0.003x^2 - 10^{-6} x^3$

For the particular item in question, the price at which x can be sold per week is given by the demand equation

p = 12 - 0.0015x.

Required: Determine the volume of sales and price at which the profit will be maximized.

Solution

To answer this question we need to establish profit as a function of quantity since the cost function is in terms of quantity produced.

From the demand equation p = 12 - 0.0015x, we have

R(x) = px = x (12 - 0.0015x)

=> P(x) = R(x) – C(x)

$= x (12 - 0.0015x) – (1000 + 6x – 0.003x^2 - 10^{-6} x^3)$

$= -1000 +6x +0.015x^2 – (10^{-6})x^3$

P(x) is maximum when $\frac{dP}{dx} = 0$, that is

$6 + 0.03x – (3x10^{-6})x^2 = 0$

Multiply both sides by -10^6 we get

$x^2 - 10{,}000x – 2{,}000{,}000 = 0$

Factorizing gives, $(x-2000)(x + 1000) = 0$

$x = 2000$ or -1000

Negative quantity has no practical meaning, thus P_{max} is when $x = 2{,}000$.

The price charged is given by the demand equation,

$p = 12 - 0.0015x$. => $p = 12 – 0.0015(2{,}000) = 12 – 3 =9$

Derivation of basic Economic Order Quantity (EOQ)

Assume a company has the following parameters in its operations:

D = Annual demand

x = Order Quantity

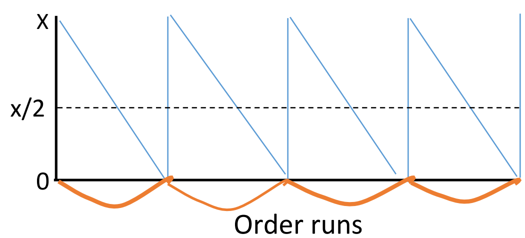
C_o = cost of ordering for one order

C_e = average carrying cost of each unit per year

Determine the order quantity that minimizes the total cost the total cost C_a. Assume that the replacement of stock is instantaneous.

Solution

Graphically this scenario is as indicated below:



Average stock = $\frac{x}{2}$

=> Total holding cost $= \frac{C_c x}{2}$

Number of orders per year = $\frac{D}{x}$

Annual ordering costs = $\frac{C_0 D}{x}$

Total cost per year $C_a(x) = \frac{C_c x}{2} + \frac{C_0 D}{x}$

The order quantity that will minimize $C_a(x)$ is given by x such that

$$\frac{d(C_a(x))}{dx} = 0 \Rightarrow \frac{d\left(\frac{C_c x}{2} + \frac{C_0 D}{x}\right)}{dx} = 0 \Rightarrow \frac{C_c}{2} - \frac{C_o D}{x^2} = 0$$

$$\Rightarrow x = \pm\sqrt{\frac{2C_0 D}{C_c}}.$$

Since negative quantity has no practical meaning, then

X_e (The Economic Order Quantity (EOQ)) = $\sqrt{\frac{2C_0 D}{C_c}}$

Application of Extremum (Optimization)

Exercize

1. A shoe manufacturer can use his plant to make either men's or women's shoes. If he makes x (in thousands of pairs) men's shoes and y (in thousands of pairs) women's shoes per week, then x and y are related by the equation,

   $$2x^2 + y^2 = 25.$$

   If the profit is \$10 on each pair of men's shoes and \$8 on women's, determine how many of each type should be made in order to maximize weekly profit.

2. The proprietor of Entebbe Beauty Salon noted that when she charges Ugx 4,000 per haircut, she gets 100 clients per week and when she charges Ugx 5,000, the number reduced to 80 per week. Assuming a Linear demand function, relating number of clients and the price per haircut, determine the marginal revenue function for the salon. Calculate the price at which the marginal revenue is zero.
3. Associated Construction company has established that the cost of erecting a building with x floors is given by a quadratic function of the form

   $$P(x) = a + bx + cx^2$$

   where a represents fixed costs such as the land costs, b represents a cost that is the same for every floor and c is the cost of structural members which are proportional to the square of the numbers of the floors.

   a) Calculate the number of floors that make the average cost per floor a minimum.

   b) Show that as the land costs increase, this optimum value of x increases.
4. Jodic interiors Ltd, has designed an inexpensive rectangular tent with no floor and one of the sides open. If the cost of the materials for the three sides and the roof is Ugx 20,000 per square metre, determine the dimensions of the shed that will minimize the cost of materials if the Tent space is 486 square metres
5. An Oil Company requires a pipeline from an offshore oilfield to a refinery which is to be built on a neighboring coast. The distance from the oilfield to a nearest point P on the coast is 20 km and the distance along the coast from P to the refinery is 50 km. From the refinery, the pipeline will go a distance x km along the coast, then it will follow a straight underwater

pipeline to the oil field. The cost per km of underwater pipeline is three times that of overland section. Find the value of x that will minimize the total cost of the pipeline.

Applications of the concept of Derivatives in Marginal Analysis

Derivatives have practical applications in many areas, both in physical and economic aspects. They are useful in analysis of rates of change, marginal costs, revenues and yield as well as in determining the greatest and least values of functions.

Marginal Analysis

Under this topic we will look at Marginal cost and Revenue, marginal productivity and optimization of business parameters such as revenues, costs and profits. In this aspect, marginal value constitutes the average change per extra item, when a very small change is made in the amount of given level of activity.

Marginal cost

Definition: The limiting value of the average cost per extra item as the number of extra items approaches zero. So MC can be conceptualized as the average cost per extra item when a very small change is made in the amount produced.

Marginal Cost = $\lim\limits_{\Delta x \to 0} \frac{\Delta c}{\Delta x}$

For example, given that a cost function has been established to be

$C(x) = 400 + 0.02x^2$, if 200 items are produced per month, the cost will be

$C(200) = 400 + 0.02\,(200)^2 = 1{,}200.$

If the manufacturer is considering a change of production levels per month to $200 + \Delta x$. this represents an increment in cost , that is

$C + \Delta C = 400 + 0.02\,(200 + \Delta x)^2 = 400 + 0.02[40{,}000 + 400\Delta x + (\Delta x)^2]$

$= 400 + 800 + 8\Delta x + 0.02(\Delta x)^2$

$\Rightarrow \Delta C = [400 + 800 + 8\Delta x + 0.02(\Delta x)^2] - 1{,}200$

$= 8\Delta x + 0.02(\Delta x)^2$

=> The average cost per extra item is $\frac{\Delta c}{\Delta x} = 8 + \Delta x$

Thus, the Marginal cost = $\lim\limits_{\Delta x \to 0} \frac{\Delta c}{\Delta x} = 8$

In general, Marginal Cost = $\lim\limits_{\Delta x \to 0} \frac{\Delta c}{\Delta x} = \frac{dc}{dx}$

Thus if the cost function is $C(x) = 0.002x^3 – 0.4x^2 + 80x + 2{,}000$, then the marginal cost function is $C'(x) = \frac{dc}{dx} = 0.006x^2 - 0.8x + 80$.

While marginal cost function is $\frac{dc}{dx}$, the average cost is C(x) (total cost of units) divided by the number of units produced, that

Average cost per item = $\frac{C(x)}{x}$ presented by the symbol $\bar{C}(x)$.

Thus the marginal cost is the average cost per additional unit, while average cost is cost per unit on average.

Example

Given a cost function of

$C(x) = 2{,}000 + 5x + 0.2x^2$

Derive the marginal and average cost functions.

Solution

The marginal cost function

$C'(x) = \frac{dc}{dx} = 5 + 0.4x$

The average cost function $\bar{C}(x) = \frac{C(x)}{x} = \frac{2{,}000+5x+0.2x^2}{x} = \frac{2{,}000}{x} + 0.2x + 5$

Marginal Revenue

Just as in the case of Marginal cost, we define marginal revenue as additional revenue per additional unit when a small increment is made in the number of items sold and its function is $R'(x) = \frac{dR}{dx}$.

Example

Given that revenue function as

$R(x) = 20x - 0.02x^2$, where x is the number of items sold,

iii) Determine the marginal revenue function
iv) What is the marginal revenue when x = 100?

Solution

iii) The Marginal Revenue function is
$R'(x) = \frac{dR}{dx} = 20 - 0.04x$
iv) When 100 items are sold , the marginal revenue, is given by $R'(100) = 20 – 0.04(100)$

21 – 4 = 16. Thus when 100 items are sold. Any small increase in sales provides an increase in revenue of 16 per item.

Marginal profit

Revenue is related to the price charged for each item sold. The law of supply and demand dictate that as the price increases the demand goes down. Also revenue is a product of products sold and the price charged. That is,

$$R(x) = xp$$

where p is the price per item and x is the number of items sold.

Example

Given that when x = 100, the demand equation is x = 2,000 – 200p. Determine the marginal revenue for the company when x = 200.

Solution

$R(x) = xp$,

but we know that $x = 2{,}000 - 200p \Rightarrow p = \frac{2{,}000 - x}{200}$

therefore,

$R(x) = x\left(\frac{2{,}000 - x}{200}\right) = 10x - 0.005x^2$

The Marginal revenue function is

$R'(x) = \frac{dR}{dx} = \frac{d(10x - 0.05x^2)}{dx} = 10 - 0.01x$

When x = 200,

$R'(x) = 10 - 0.01(200) = 8$

Profit function

Profit function P(x) is given as

$P(x) = R(x) - C(x)$,

where R(x) and C(x) are revenue and cost functions respectively.

Therefore, the Marginal profit

$P'(x) = R'(x) - C'(x)$

Example

Given the demand function as p + 0.2x = 110 and cost function as c(x) = 2,000 +10x, determine the marginal profits when 100 units are sold and when 300 units are sold. Comment of the two results.

Solution

$R(x) = xp = x(110 – 0.2x) = 110x – 0.2x^2$

=> $P(x) = (110x – 0.2x^2) – (2,000 + 10x) = 100x – 0.2x^2 – 2,000$

Therefore, the Marginal Profit function, $P^{/}(x) = \frac{d(100x-0.2x^2-2,000)}{dx} = 100 – 0.4x$

When x =100, MP = 100 -0.4(100) = 60, and when x = 300, MP = 100 – 0.4 (300) = -20.

When producing 100 units a small increase in production will yield an additional profit of 60 per unit while when producing at 300, and additional increment in the units will lead to a loss of 20 per additional unit.

Example

The demand equation for Laptops at ICTcom Uganda Limited, is given by the curve $p = 300e^{-\frac{x}{20}}$, where **x** units are sold at a price of p (in dollars) each. If the ICTcom has a fixed cost of 500 (in dollars) and a variable cost of 20 (in dollars) per unit, find its

(i) Marginal Revenue function

$R(x) = px = 300xe^{-x/20}$

Using product rule of differentiation, marginal revenue function

$R^{/}(x) = 300[\frac{-x}{20}e^{-x/20} + e^{-x/20}] = 300e^{-x/20}[1 - \frac{x}{20}] = 15e^{-x/20}[20 – x]$

(ii) Marginal profit function

Profit function

$P(x) = R(x) – C(x)$

=> Marginal profit function is $P^{/}(x) = R^{/}(x) – C^{/}(x)$

$C(x) = 20x + 500$ => $C^{/}(x) = 20$

$\therefore P^{/}(x) = 15e^{-x/20}[20 – x] – 20$

Marginal productivity

Productivity is the amount of output from a company per given period. Consider a situation when a company has h as the number hours available per week and that this produces x amount of output. It can be said that

$x = f(h)$

If the amount of labour is given an increment of Δh then

$\Delta x = f(h + \Delta h) - f(h)$

The average increment with respect to h is

$$\frac{\Delta x}{\Delta h} = \frac{f(h+\Delta h) - f(h)}{\Delta h}$$

Thus the amount of average additional production per extra unit of labour corresponding to a given increase Δh is the Marginal productivity given as

$$\frac{dx}{dh} = \lim_{\Delta h \to 0} \left(\frac{f(h+\Delta h) - f(h)}{\Delta h} \right)$$

Thus marginal productivity of labour measures the increase in productivity per additional unit of labour, when a small change in the amount of labour is employed.

Marginal yield

Yield in financial terms means the amount earned from an investment. This is not to be confused with return on investment which is the total amount of profit generated divided by the amount invested.

If y is the yield function and s the investment, then,

$$\text{Marginal Yield} = \frac{dy}{ds}$$

Exercise

5. Calculate the marginal cost functions for the following functions:
   d) $C(x) = 200 + 2x - 0.06x^2 + 0.0002x^3$
   e) $C(x) = 10^{-4} x^3 - 0.02x^2 40x + 400$
   f) $C(x) = (\ln 4)x^3 + 50$
6. Given a revenue function,
   $R(x) = 100x - x^3(1 + \sqrt{x})$,
   find the marginal revenue function.

7. If the demand equation is $12p + 3x + 0.012x^2 = 720$,
   c) find the marginal revenue when $p = 15$
   d) Find the value of x that makes $P^/(x) = 0$ and calculate the corresponding profit.
8. If the cost function is of the form, $Ax^2 + Bx + C$, at what value of x is the marginal cost equal to the average cost $\bar{C}(x)$.

Marginal Profit Example

A shoe manufacturer can use his plant to make either men's or women's shoes. If he makes x (in thousands of pairs) men's shoes and y (in thousands of pairs) women's shoes per week, then x and y are related by the equation,

$$2x^2 + y^2 = 25.$$

If the profit is \$10 on each pair of shoes, calculate the marginal profit with respect to x when x = \$8.